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A syntax-based approach to measuring the degree of inconsistency for belief bases

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ABSTRACT

Measuring the degree of inconsistency of a belief base is an important issue in many real-world applications. It has been increasingly recognized that deriving syntax sensitive inconsistency measures for a belief base from its minimal inconsistent subsets is a natural way forward. Most of the current proposals along this line do not take the impact of the size of each minimal inconsistent subset into account. However, as illustrated by the well-known Lottery Paradox, as the *size* of a minimal inconsistent subset increases, the *degree* of its inconsistency decreases. Another lack in current studies in this area is about the role of free formulas of a belief base in measuring the degree of inconsistency. This has not yet been characterized well. Adding free formulas to a belief base can enlarge the set of consistent subsets of that base. However, consistent subsets of a belief base also have an impact on the syntax sensitive normalized measures of the degree of inconsistency, the reason for this is that each consistent subset can be considered as a distinctive plausible perspective reflected by that belief base, whilst each minimal inconsistent subset projects a distinctive view of the inconsistency. To address these two issues, we propose a normalized framework for measuring the degree of inconsistency of a belief base which unifies the impact of both consistent subsets and minimal inconsistent subsets. We also show that this normalized framework satisfies all the properties deemed necessary by common consent to characterize an intuitively satisfactory measure of the degree of inconsistency for belief bases. Finally, we use a simple but explanatory example in requirements engineering to illustrate the application of the normalized framework.

1. Introduction

Interest in measuring inconsistency for belief bases has grown rapidly in recent years. This interest is driven by the desire to understand how measures of inconsistency can be formalized (e.g., [1–3]), and by the practical needs of real-world applications (e.g., [4–6]). A number of proposals for measuring the degree of inconsistency of a belief base have been presented, including the maximal η-consistency [7,8], n-consistency and n-probability [9], measures based on variables or paraconsistent models [1,10–13], measures based on minimal inconsistent subsets [14,3], and the Shapley Inconsistency Value [2].

In particular, the measures based on minimal inconsistent subsets are attractive for syntax sensitive conflict resolution in some applications such as requirements engineering [3]. As a matter of fact, for conflict resolution, the minimal inconsistent

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subsets of a belief base can be considered as the purest expression of inconsistency, since an agent needs to remove only one formula from each minimal inconsistent subset to restore consistency [15]. Following this viewpoint, several approaches to measuring inconsistency based on minimal inconsistent subsets have been proposed. The scoring function [14] and the MI measure [3] are the most representative methods of such approaches to measuring the degree of inconsistency. The scoring function assigns each subset of a belief base the number of minimal inconsistent subsets of the base that would be eliminated if the subset was removed from that belief base [14]. The degree, or relative amount of inconsistency of a belief base is captured by scoring function values of all the non-empty subsets of that belief base together. In contrast, the MI measure views the total number of minimal inconsistent subsets of a belief base as a measure of the amount of inconsistency of that belief base [3]. The common idea of the scoring function and the MI measure is to focus on counting the number of minimal inconsistent subsets of a belief base.

Conspicuously, these two methods do not make any explicit distinction between two minimal inconsistent subsets with different sizes. Moreover, the MinInc property presented in [3] states that each minimal inconsistent subset brings the same amount of inconsistency. However, as the cardinality of a minimal inconsistent subset increases, its inconsistency becomes more tolerable [7,3], i.e., the bigger the size of a minimal inconsistent subset, the smaller the degree of its inconsistency. To illustrate this in an informal way, let us consider the well-known lottery paradox which motivated Knight proposing his approach [7]. The lottery paradox presented in [16] considered an n-ticket lottery scenario, known to be fair and to have exactly one winner. It is rational to accept that *for any individual ticket i, ticket i will not win*, since the probability of ticket i being the winner cannot exceed a given, high enough threshold due to the fairness of the lottery. Then $K(n) = \{\neg w_1, \ldots, \neg w_n, w_1 \vee \cdots \vee w_n\}$ is a minimal inconsistent belief base (we will define this concept carefully later) for the lottery, where for each i, w_i asserts that ticket i is the winner. Intuitively, if there are a sufficiently large number of tickets in the lottery, say 1 million, the belief base $K(n)$ is *almost consistent*(in our case only 1 in 1 million is wrong), whereas $K(n)$ is *highly inconsistent* if there are, say, only three tickets. Therefore, considering only the total number of minimal inconsistent subsets is not enough for precisely capturing the degree of inconsistency of a belief base. The size of each minimal inconsistent subset should be considered.

In addition to this, the role of free formulas of a belief base (i.e., formulas not belonging to any minimal inconsistent subset of a belief base) in measuring the degree of inconsistency of that belief base has not yet been characterized well. According to the viewpoint that minimal inconsistent subsets are the purest form of inconsistency, free formulas of a belief base have nothing to do with the conflicts of that belief base because free formulas do not belong to any minimal inconsistent of that belief base. To uphold this viewpoint, the Free Formula Independence property presented in [2,3] requires that adding a free formula to a belief base should not change the inconsistency measure of that base.

However, the Free Formula Independence property does not capture the impact of free formulas on the degree of inconsistency of a belief base well for syntax-based measures as well as for model-based or variable-based measures. Roughly speaking, model-based or variable-based inconsistency measures are often based on some paraconsistent models, which allow us to assign an inconsistent truth value (i.e., both true and false) to variables involved in inconsistency. Then the degree of inconsistency of a belief base is often captured by the *normalized minimum number of inconsistent truth values* in such a model. A free formula not belonging to any minimal inconsistent subset cannot guarantee that it is free from confliction with any other formulas. To illustrate this, consider $K = M \cup \{\neg b\}$, where $\neg b$ is a free formula of K and $M = \{a \wedge \neg a \wedge b\}$ is the minimal inconsistent subset of K. Clearly, M conveys contradictory information about a alone, but K conveys contradictory information about b as well as that about a. So variable b must have been involved in inconsistent values in such paraconsistent models. In such cases, adding a free formula to a belief base increases the degree of inconsistency of that belief base. For example, as pointed out in [17], the $\mathsf{I}_{\mathsf{LP_m}}$ inconsistency measure presented in [2] allows some free formulas of a belief base to increase the existing conflicts in minimal inconsistent subsets of that belief base. On the other hand, for a given belief base, if we add a formula consisting of new variables that do not appear in formulas of that belief base, then the minimum number of inconsistent truth values will not change since the enlarged belief base cannot convey contradictory information about these new variables. But the degree of inconsistency of that base will be diluted by normalization since the new variables enlarge the set of all variables. For example, consider $\mathsf{I}_{\mathsf{LP_m}}$ again, $\mathsf{I}_{\mathsf{LP_m}}(\{a \wedge \neg a \wedge \neg b\}) = \frac{1}{2}$. After adding $c \wedge d$ to $\{a \wedge \neg a \wedge \neg b\}$, $\mathsf{I}_{\mathsf{LP_m}}(\{a \wedge \neg a \wedge \neg b, c \wedge d\}) = \frac{1}{4}$. In summary, in the context of model-based or variable-based inconsistency measures, adding free formulas can either strengthen or weaken the degree of inconsistency. The impact of free formulas of a belief base on the degree of that belief base depends on literals of free formulas. Therefore, model-based or variable-based inconsistency measures may not provide support for the Free Formula Independence property.

Even if we consider syntax-based or syntax sensitive proposals for measuring the degree of inconsistency, the Free Formula Independence property presents problems. It is again not an undisputed property to characterize measures for the degree of inconsistency of belief bases. Intuitively, in syntax sensitive application domains, as the number of free formulas of a belief base increases the degree of its inconsistency will be diluted if its set of minimal inconsistent subsets remains the same. This discussion, nevertheless, is still in harmony with the viewpoint that minimal inconsistent subsets are the purest form of inconsistency. This can be explained by pointing out that each minimal inconsistent subset projects a distinctive viewpoint of the inconsistency, whilst each consistent subset projects a different but completely consistent viewpoint of the same base. It is again intuitive that how inconsistent a belief base is should depend on how much distinctive inconsistent viewpoints account for all the distinctive (inconsistent or plausible) viewpoints reflected by that belief base. Therefore, the degree of inconsistency of a belief base should be influenced by both the consistent subsets as well as the minimal inconsistent subsets. In this sense, adding a free formula will certainly enlarge the set of consistent subsets. Correspondingly, adding or removing free formulas from a belief base may decrease or increase the degree of inconsistency of that belief base. This intuition is also partially supported by most of the inconsistency measures based on variables. To illustrate, let us consider the inconsistency measure $\mathsf{I}_{\mathsf{LP_m}}$ presented in [2], which is the normalized minimum number of inconsistent truth values in the $\mathsf{LP_m}$ models [18]. For any belief base K, $\mathsf{I}_{\mathsf{LP_m}}(K \cup \{\alpha\}) < \mathsf{I}_{\mathsf{LP_m}}(K)$ if α is a free formula of $K \cup \{\alpha\}$ and α does contain only variables that do not appear in formulas of K.

To address these issues, in this paper, we first present a framework for measuring the degree of inconsistency for belief bases, integrating the impact of both consistent and minimal inconsistent subsets. We then discuss the intuitive properties that such measures should satisfy. Our study shows that a measure instantiated by our framework gives more intuitive measurements for the degree of inconsistency for belief bases than other measures.

The rest of this paper is organized as follows: In the next section, we give some necessary notations about minimal inconsistent subsets of a belief base. In Section 3 we briefly recall some closely related work. In Section 4, we define a framework for measuring the degree of inconsistency for a belief base based on both minimal inconsistent subsets and consistent subsets. In Section 5, we propose a set of properties to characterize measures of the degree of inconsistency for belief bases. Then we give some instantiated measures in Section 6. In Section 7, we present a case study to illustrate the application of our approach in the domain of requirements engineering. In Section 8, we compare our measures with some closely related work. Finally, we conclude this paper in Section 9.

2. Preliminaries

Throughout this paper, we use a finite propositional language. Let $\mathcal{P}$ be a finite set of propositional symbols and $\mathcal{L}$ a propositional language built from $\mathcal{P}$ under connectives $\{\neg, \wedge, \vee, \rightarrow\}$. We use $a, b, c, \ldots$ to denote the propositional variables , and $\alpha, \beta, \gamma, \cdots$ to denote the propositional formulas.

A belief base K is a finite set of propositional formulas. We use $\mathcal{K}_{\mathcal{L}}$ to denote the set of belief bases definable from formulas of the language $\mathcal{L}$. A belief base K is inconsistent if there is a formula α such that $K \vdash \alpha$ and $K \vdash \neg\alpha$, where $\vdash$ is the classical consequence relation. We abbreviate $\alpha \wedge \neg\alpha$ as $\perp$ if there is no confusion. Then an inconsistent belief base K is denoted by $K \vdash \perp$. Moreover, an inconsistent belief base K is called a *minimal inconsistent set* (or *minimal inconsistent belief base*) if none of its proper subsets is inconsistent. If $K' \subseteq K$ and K' is a minimal inconsistent set, then we call K' a *minimal inconsistent subset* of K. We use $\mathsf{MI}(K)$ to denote a set of the minimal inconsistent subsets of K, i.e.,

$$\mathsf{MI}(K) = \{K' \subseteq K | K' \vdash \perp \text{ and } K'' \nvdash \perp \text{ for all } K'' \subset K'\}.$$

The minimal inconsistent subsets can be considered as the purest form of inconsistency for syntax sensitive conflicts resolution, since one has just to remove one formula from each minimal inconsistent subset in such cases to make it consistent [15]. In contrast, each consistent subset of a belief base may be considered as one of the plausible views of that belief base. This also accords with the viewpoint of the presence of inconsistency as a result of information pollution by wrong data such as [19], which insists on consistent subsets are meaningful, despite the pollution. We use $\mathsf{CN}(K)$ to denote the set of the consistent subsets of K, i.e.,

$$\mathsf{CN}(K) = \{\emptyset \subset K' \subseteq K | K' \nvdash \perp\}.$$

We call a formula of K a *free formula* of K if this formula does not belong to any minimal inconsistent subset of K [2]. That is, the free formulas of K have nothing to do with the minimal inconsistent subsets of K. We use $\mathsf{FREE}(K)$ to denote the set of free formulas of K, i.e.,

$$\mathsf{FREE}(K) = \{\alpha \in K | \alpha \notin M \text{ for all } M \in \mathsf{MI}(K)\}.$$

As illustrated by the following lemma, adding free formulas to a belief base can enlarge the set of consistent subsets of that belief base.

Lemma 2.1. *Let K be a belief base and α a formula not belonging to K. If α is a free formula of $K \cup \{\alpha\}$, then*

$$\mathsf{CN}(K \cup \{\alpha\}) = \mathsf{CN}(K) \cup \{\{\alpha\}\} \cup \{M \cup \{\alpha\} | M \in \mathsf{CN}(K)\}.$$

Proof. Let $K' \in \mathsf{CN}(K \cup \{\alpha\})$, then $\alpha \notin K'$ or $\alpha \in K'$.

- If $\alpha \notin K'$, then $K' \subseteq K$ and $K' \nvdash \perp$. So $K' \in \mathsf{CN}(K)$.
- If $\alpha \in K'$, then $K' = \{\alpha\}$ or $\{\alpha\} \subset K'$. Further, if $\{\alpha\} \subset K'$, then $\emptyset \subset K' \setminus \{\alpha\} \subseteq K$ and $K' \setminus \{\alpha\} \nvdash \perp$. So, $K' \setminus \{\alpha\} \in \mathsf{CN}(K)$. $\square$

Example 2.1. Consider $K_1 = \{a, b, \neg b\}$. Then

$$\mathsf{MI}(K_1) = \{\{b, \neg b\}\},$$
$$\mathsf{CN}(K_1) = \{\{a\}, \{b\}, \{\neg b\}, \{a, b\}, \{a, \neg b\}\}.$$
$$\mathsf{FREE}(K_1) = \{a\}.$$

3. Related work

In this section, we give a brief introduction to current inconsistency measures illustrating the roles of minimal inconsistent subsets and free formulas, as well as properties to characterize inconsistency measures, respectively.

3.1. The scoring function

The scoring function presented in [14] may be considered as an early representative measure of the degree of inconsistency of a belief base defined from minimal inconsistent subsets of that belief base. Roughly speaking, the idea of scoring functions for a belief base focuses on measuring the contribution made by each subset of that belief base to the inconsistency. For a belief base K, a scoring function S is defined from 2^K (the power set of K) to the natural numbers so that for any subset of K, denoted K', $S(K')$ gives the number of minimal inconsistent subsets of K that would be eliminated if subset K' was removed from K. That is,

$$S(K') = |\mathsf{MI}(K)| - |\mathsf{MI}(K - K')| \quad \text{for all } K' \subseteq K.$$

As such, belief bases with the same size can be compared using their scoring functions so that an ordering relation, which means *more inconsistent than*, over these belief bases can be defined [14]. Assume that K_1 and K_2 are of the same cardinality, S_1 and S_2 are the scoring functions for K_1 and K_2 respectively, then $S_1 \leq S_2$ holds if and only if there is a bijection $f : 2^{K_1} \mapsto 2^{K_2}$ such that the following condition can be satisfied:

$$S_1(K') \leq S_2(f(K')) \quad \text{for all } K' \subseteq K_1.$$

We say K_2 is more inconsistent than K_1 if and only if $S_1 \leq S_2$.

Example 3.1. Consider $K_1 = \{a, b, \neg b\}$ and $K_2 = \{a \wedge \neg a, b, \neg b\}$. Then $\mathsf{MI}(K_1) = \{\{b, \neg b\}\}$ and $\mathsf{MI}(K_2) = \{\{a \wedge \neg a\}, \{b, \neg b\}\}$. So,

$$\begin{array}{llll}
S_1(K_1) = 1, & S_1(\{a, b\}) = 1, & S_1(\{b, \neg b\}) = 1, & S_1(\{a, \neg b\}) = 1, \\
S_1(\{b\}) = 1, & S_1(\{\neg b\}) = 1, & S_1(\{a\}) = 0, & S_1(\emptyset) = 0, \\
S_2(K_2) = 2, & S_2(\{a \wedge \neg a, b\}) = 2, & S_2(\{a \wedge \neg a, \neg b\}) = 2, & S_2(\{b, \neg b\}) = 1, \\
S_2(\{b\}) = 1, & S_2(\{\neg b\}) = 1, & S_2(\{a \wedge \neg a\}) = 1, & S_2(\emptyset) = 0.
\end{array}$$

Evidently, $S_1 < S_2$. Therefore, we may say K_2 is more inconsistent than K_1.

Note that the scoring function uses $2^{|K|}$ values rather than a single value to capture the degree of inconsistency of a belief base. Moreover, the ordering relation *more consistent than* is defined only between any two belief bases with the same size. This makes it difficult to apply the scoring function based approaches to comparing two arbitrary inconsistent belief bases. For example, as in the lottery paradox, we cannot derive that $\{\neg w_1, \neg w_2, w_1 \vee w_2\}$ is more inconsistent than $\{\neg w_1, \neg w_2, \ldots, \neg w_n, w_1 \vee w_2 \vee \cdots \vee w_n\}$(where n is large enough) by using their scoring functions, although this comparison is intuitive.

We must point out that the scoring function implicitly supports the intuition that as the number of free formulas increases, the degree of inconsistency of a belief base becomes smaller. To illustrate, consider $K_3 = \{\neg w_1, \neg w_2, w_1 \vee w_2\} \cup \{w_3, \ldots, w_n\}$ and $K_4 = \{\neg w_1, \neg w_2, \ldots, \neg w_n, w_1 \vee \cdots \vee w_n\}$. Suppose that n is much larger than 2, then intuitively $\{\neg w_1, \neg w_2, w_1 \vee w_2\}$ is *more inconsistent than* K_4, as illustrated by the lottery paradox. On the other hand, note that $\{w_3, \ldots, w_n\}$ is the set of free formulas of K_3. Let S_3 and S_4 be the scoring functions for K_3 and K_4, respectively, then clearly $S_3 < S_4$. So, K_4 *is more inconsistent than* K_3. Therefore, we may conclude that $\{\neg w_1, \neg w_2, w_1 \vee w_2\}$ is *more inconsistent than* K_3.

3.2. The MI inconsistency measure

The MI inconsistency measure and the family of the MinInc inconsistency values presented in [3] are the most recent representative measures defined from minimal inconsistent subsets. The MI inconsistency measure aims to measure the amount of inconsistency of a belief base in terms of minimal inconsistent subsets of that belief base, and the family of the MinInc inconsistency values focus on measuring the inconsistency value of each formula belonging to an inconsistent belief base through minimal inconsistent subsets. Given a belief base K, the MI inconsistency measure for K, denoted $\mathsf{I}_{\mathsf{MI}}(K)$, is defined as the number of minimal inconsistent subsets of K, i.e.,

$$\mathsf{I}_{\mathsf{MI}}(K) = |\mathsf{MI}(K)|.$$

Obviously, according to MI inconsistency measure, the number of minimal inconsistent subsets of a belief base can be used to capture the inconsistency of that belief base. However, neither the MI inconsistency measure nor the normalized MI inconsistency measure $\left(\frac{\mathsf{I}_{\mathsf{MI}}(K)}{2^{|K|}} \text{ or } \frac{\mathsf{I}_{\mathsf{MI}}(K)}{2^{|K|}-1}\right)$ makes a distinction in the degree of inconsistency between any two belief bases with the same size and the same number of minimal inconsistent subsets. To illustrate this, consider K_3 and K_4 again. Evidently, $\mathsf{I}_{\mathsf{MI}}(K_3) = \mathsf{I}_{\mathsf{MI}}(K_4) = 1$. *This is in contrast to the comparison result given by the scoring function approach above.*

3.3. Maximal η-consistency

The MI inconsistency measure does not consider the sizes of different minimal inconsistent subsets in the degree of inconsistency. However, as illustrated by the lottery paradox, the degree of inconsistency of a minimal inconsistent set becomes smaller or more tolerable as the size of a minimal inconsistent set increases [7]. The method of maximal η-consistency presented in [7] provides direct support for this intuition. Roughly speaking, this method is based on a probability function P over $\mathcal{L}$ presented in [20,21], which satisfies:

- if $\models \alpha$, then $P(\alpha) = 1$,
- if $\models \neg(\alpha \wedge \beta)$, then $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$.

Then a belief base K is η-consistent ($0 \leq \eta \leq 1$) if there is a probability function P such that $P(\alpha) \geq \eta$ for all $\alpha \in K$. Furthermore, K is maximally η-consistent if η is maximal. Evidently, maximal 1-consistency corresponds to complete consistency, and maximal 0-consistency corresponds to the explicit presence of a contradiction, i.e., the explicit presence of a contradictory formula in a belief base [7]. Intuitively, we may use $1 - \eta$ to capture the inconsistency of K if K is maximally η-consistent.

In particular, it has also been shown in [7] that a minimal inconsistent belief base M is maximally $\left(\frac{|M|-1}{|M|}\right)$-consistent [7]. Recall the n-ticket lottery paradox again, $K(n) = \{\neg w_1, \ldots, \neg w_n, w_1 \vee \cdots \vee w_n\}$ is maximally $\frac{n}{n+1}$-consistent, which explains why $K(n)$ is highly inconsistent if there are few tickets in the lottery, whilst $K(n)$ is nearly consistent if there are millions of tickets.

Compared to the scoring function and the MI inconsistency measure, maximal η-consistency considers the size of a minimal inconsistent belief base explicitly. On the other hand, a lower bound of η for an inconsistent belief base K that contains no contradiction is $\frac{|K'|-1}{|K|}$, where K' is a smallest minimal inconsistent subset of K [7]. This implies that the lower bound of η for K depends on the size of K as well as the size of the smallest minimal inconsistent subsets of K. So, the lower bound of η will decrease if we enlarge a belief base by adding free formulas. In this sense, the maximal η-consistency also partially supports the intuition that as the number of free formulas increases the degree of inconsistency of a belief base becomes smaller.

A close work to the maximal η-consistency is the n-consistency and n-probability presented in [9]. As pointed out in [9], the semantic notion of n-probability is similar to η-consistency. Roughly speaking, n-consistency focuses on counting the number of formulas needed for deriving a contradiction. A theory T is strictly n-consistent if each subset of I of cardinality n is consistent and at least one subset of cardinality $n+1$ is inconsistent. So, a minimal inconsistent subset of cardinality n is strictly $(n-1)$-consistent. This implies that the n-consistency also provides support for taking the size of each minimal inconsistent subset into account.

3.4. Inconsistency measures based on variables

We consider a more general but representative inconsistency measure based on variables presented in [2], denoted $\mathsf{I}_{\mathsf{LP_m}}$, which is defined from $\mathsf{LP_m}$ models of a belief base within Priest's logic of paradox [18]. Roughly speaking, Priest's logic of paradox (LP for short) aimed to provide three-valued models for classically inconsistent sets of formulas by using the set $\{\mathsf{T}, \mathsf{F}, \mathsf{B}\}$ of truth values, in which the third truth value B is considered intuitively as both true and false and taken as a designated value together with T. An interpretation ω for $\mathsf{LP_m}$ models maps each propositional variable to one of the three truth values T, F, B such that

- $\omega(\neg\alpha) = \mathsf{B}$ if and only if $\omega(\alpha) = \mathsf{B}$,
- $\omega(\neg\alpha) = \mathsf{T}$ if and only if $\omega(\alpha) = \mathsf{F}$,
- $\omega(\neg\alpha) = \mathsf{F}$ if and only if $\omega(\alpha) = \mathsf{T}$,
- $\omega(\alpha \wedge \beta) = \min_{\leq t}\{\omega(\alpha), \omega(\beta)\}$,
- $\omega(\alpha \vee \beta) = \max_{\leq t}\{\omega(\alpha), \omega(\beta)\}$,

where $\mathsf{F} <_t \mathsf{B} <_t \mathsf{T}$. Then the set of models of a formula α is defined as $\mathsf{Mod}_{\mathsf{LP}}(\alpha) = \{\omega | \omega(\alpha) \in \{\mathsf{T}, \mathsf{B}\}\}$. The inconsistency measure $\mathsf{I}_{\mathsf{LP_m}}$ is the normalized minimum number of inconsistent truth values in models of K [2], i.e.,

$$\mathsf{I}_{\mathsf{LP_m}}(K) = \frac{\min_{\omega \in \mathsf{Mod}_{\mathsf{LP}}(K)}(|\omega!|)}{|\mathcal{P}|},$$

where $\omega! = \{x \in Var(K) | \omega(x) = \mathsf{B}\}$.

As pointed out in [17], adding a free formula to a belief base may change the degree of its inconsistency. To illustrate this, consider $K = \{a \wedge \neg a \wedge \neg b\}$. Then $\mathsf{I}_{\mathsf{LP_m}}(K \cup \{c\}) = \frac{1}{3} < \mathsf{I}_{\mathsf{LP_m}}(K) = \frac{1}{2} < \mathsf{I}_{\mathsf{LP_m}}(K \cup \{b\}) = 1$. This signifies that free formulas are not necessarily free or independent from inconsistency in such variable-based inconsistency measures.

3.5. Properties to characterize inconsistency measures

On the other hand, the question of how to characterize a desirable inconsistency measure, especially in terms of minimal inconsistent subsets, is still under development, even though Consistency, Monotony, Free Formula Independence, Dominance, and Normalization have been presented in [2,3] as fundamental properties of a basic inconsistency measure for a whole belief base. For instance in [2], a basic inconsistency measure for belief bases, denoted I, needs to satisfy the following five properties:

- *Consistency* : $\mathsf{I}(K) = 0$ if and only if K is consistent.
- *Monotony*: $\mathsf{I}(K \cup K') \geq \mathsf{I}(K)$.
- *Free Formula Independence*: If $\alpha \in \mathsf{FREE}(K \cup \{\alpha\})$, then $\mathsf{I}(K \cup \{\alpha\}) = \mathsf{I}(K)$.
- *Dominance*: If $\alpha \vdash \beta$ and $\alpha \nvdash \perp$, then $\mathsf{I}(K \cup \{\alpha\}) \geq \mathsf{I}(K \cup \{\beta\})$.
- *Normalization*: $0 \leq \mathsf{I}(K) \leq 1$.

The property of Consistency requires that an intuitive inconsistency measure should assign null to a consistent belief base. This can be considered as one of the most natural constraints to a desirable inconsistency measure for belief bases.

The property of Normalization is not mandatory. It is added only for simplification purpose [2]. Roughly speaking, the Normalization property provides the lower bound 0 and the upper bound 1 for a normalized inconsistency measure. Generally, the bounds of an inconsistency measure should be given clear meanings in the context of application domains. For example, as mentioned above, in the context of maximal η-consistency, maximal 1-consistency corresponds to complete consistency, and maximal 0-consistency corresponds to the explicit presence of a contradiction, i.e., the explicit presence of a contradictory formula in a belief base [7]. Note that the Consistency property explains the meaning of the lower bound 0. We also need a new property to clarify explicitly the meaning of the upper bound 1.

As explained in [2], the Dominance property states that logically stronger formulas should bring (potentially) more conflicts. Precisely speaking, logically stronger formulas may bring (potentially) more *absolute* amount of inconsistency in the context of variable-based or model-based inconsistency measurements. In other words, logically stronger formulas may bring more variables assigned to inconsistent truth values. However, we cannot ensure that logically stronger formulas always bring no information other than conflicts. This implies that logically stronger formulas cannot ensure more *relative* amount of inconsistency or higher degree of inconsistency in the context of variable-based or model-based characterization of inconsistency. In this sense, the Dominance property is inappropriate to characterizing variable-based or model-based measures for the degree of inconsistency. To illustrate this, consider $\{\neg a, a\}$ and $\{\neg a, a \wedge b\}$. Obviously, $a \wedge b$ is logically stronger than a. Besides conveying the contradictory information about a (together with $\neg a$), $a \wedge b$ conveys meaningful information about b. Intuitively, $\{\neg a, a \wedge b\}$ is less inconsistent than $\{\neg a, a\}$ in the context of variable-based or model-based characterization of inconsistency. Actually, variable-based or model-based inconsistency measures accord with this intuition and may not provide support for the Dominance property. To illustrate this, consider $K \cup \{b\} = \{a \wedge \neg a, \neg b, b, c\}$ and $K \cup \{b \wedge d\} = \{a \wedge \neg a, \neg b, b \wedge d, c\}$. Then $\mathsf{I}_{\mathsf{LP_m}}(K \cup \{b \wedge d\}) = \frac{1}{2} < \mathsf{I}_{\mathsf{LP_m}}(K \cup \{b\}) = \frac{2}{3}$.

On the other hand, within the context of syntax sensitive characterization of inconsistency, the Dominance property is also inappropriate for characterizing inconsistency measures based on minimal inconsistent subsets. Inconsistency measures based on minimal inconsistent subsets consider minimal inconsistent subsets as the purest forms of inconsistencies. This viewpoint states that the inconsistency of a belief base is conveyed by only the set of all the minimal inconsistent subsets of that belief base. Note that within the context of syntax sensitive characterization of inconsistency, each formula belonging to a belief base has special meanings and is viewed as a smallest unit. Therefore, to characterize the inconsistency for a belief base, the number of minimal inconsistent subsets and the size of each minimal inconsistent subset rather than the variables that constitute the formulas involved in the inconsistency need to be considered. However, the minimal inconsistent subset is of syntax sensitivity. We cannot ensure that logically stronger formulas result in a stronger set of minimal inconsistent subsets. This makes the Dominance property inappropriate within the context of syntax sensitive characterization of inconsistency based on minimal inconsistent subsets. To illustrate this, consider two minimal inconsistent belief bases $K = \{a \wedge \neg b, \neg a\}$ and $K' = \{a \wedge \neg b, \neg a \wedge b\}$. Within the context of variable-based or model-based characterization of inconsistency, it is intuitive to consider that K' has more conflicts, since K' conveys contradictory information about b as well as that about

a, whilst K conveys only contradictory information about a. But this type of intuition does not necessarily hold within the context of characterization of inconsistency based on minimal inconsistent subsets, because we consider each formula rather than variables appearing in the formula as the smallest unit. In such a syntax sensitive context, either K or K' conveys one conflict between two self-consistent formulas. Then it is intuitive to consider that K has the same amount inconsistency as that of K', although $\neg a \wedge b$ is logically stronger than $\neg a$. Note that the proposal of maximal η-consistency also supports this intuition, since either K or K' is maximally $\frac{1}{2}$-consistent.

Generally, conflicts involving β implied by some minimal inconsistent subsets of $K \cup \{\beta\}$ will be represented by some minimal inconsistent subsets with the same size as, or smaller than, of $K \cup \{\alpha\}$, if α is logically stronger than β. Allowing for the syntax sensitivity of minimal inconsistent subsets, it is difficult to establish a meaningful correspondence between $\mathsf{MI}(K \cup \{\alpha\})$ and $\mathsf{MI}(K \cup \{\beta\})$, except that $\forall M \in \mathsf{MI}(K \cup \{\beta\})$, if $\beta \in M$, then $\exists M' \subseteq M \cup \{\alpha\} - \{\beta\}$ s.t. $M' \in \mathsf{MI}(K \cup \{\alpha\})$. In this sense, to satisfy the Dominance property, the measure for a belief base that has one minimal inconsistent subset with a smaller size needs to be greater than the measure for a belief base that has at least one minimal inconsistent subsets with a larger size. To illustrate this, consider $K(n) = \{a_1, a_1 \wedge a_2, \ldots, a_1 \wedge a_n, \neg b\}$. Then

$$\mathsf{MI}(K(n) \cup \{\neg a_1 \vee b\}) = \{M_1, \ldots, M_n\},$$

where

$$\begin{aligned}
M_1 &= \{a_1, \neg a_1 \vee b, \neg b\},\\
M_2 &= \{a_1 \wedge a_2, \neg a_1 \vee b, \neg b\},\\
&\ \ \vdots\\
M_n &= \{a_1 \wedge a_n, \neg a_1 \vee b, \neg b\}.
\end{aligned}$$

Let α_n be $a_1 \wedge a_2 \wedge \cdots \wedge a_n \wedge (\neg a_1 \vee b)$, then

$$\mathsf{MI}(K(n) \cup \{\alpha_n\}) = \{\{a_1 \wedge a_2 \wedge \cdots \wedge a_n \wedge (\neg a_1 \vee b), \neg b\}\}.$$

To uphold the Dominance property, the evaluation assigned to a set of one 2-size minimal inconsistent subset $\mathsf{MI}(K(n) \cup \{\alpha_n\})$ should be no less than that assigned to a set of n 3-size minimal inconsistent subsets $\mathsf{MI}(K(n) \cup \{\neg a_1 \vee b\})$ for any $n > 0$. However, it seems to be difficult to satisfy this constraint if we use numerical measurements. A possible way to handle this issue is to assign a sovereign importance to minimal inconsistent subsets of smaller size. For example, within the context of maximal η-consistency, it has been pointed out in [7] that $\frac{|M|-1}{|K|}$ can be considered as a lower bound of η for an inconsistent knowledge base K containing no contradiction, where M is the smallest minimal inconsistent subset. However, this may result in that minimal inconsistent subsets of larger size play no explicit role in measuring the degree of inconsistency. Intuitively, if a measure for inconsistency is based on integration of all the minimal inconsistent subsets, it is difficult to provide support for the Dominance property. Indeed, as the most representative measures, both the scoring function and the MI inconsistency measure do not satisfy the Dominance property. To illustrate this, consider the following example.

Example 3.2 (*Counterexample about the Dominance Property*). Consider $K = \{a, a \wedge c, \neg b\}$. Suppose that $\alpha = a \wedge c \wedge (\neg a \vee b)$ and $\beta = \neg a \vee b$. Then $\alpha \vdash \beta$ and $\alpha \nvdash \perp$. The minimal inconsistent subsets of $K \cup \{\alpha\}$ and $K \cup \{\beta\}$ are given as follows, respectively.

$$\begin{aligned}
\mathsf{MI}(K \cup \{\alpha\}) &= \{\{a \wedge c \wedge (\neg a \vee b), \neg b\}\},\\
\mathsf{MI}(K \cup \{\beta\}) &= \{\{a, \neg a \vee b, \neg b\}, \{a \wedge c, \neg a \vee b, \neg b\}\}.
\end{aligned}$$

Clearly,

$$\mathsf{I}_{\mathsf{MI}}(K \cup \{\alpha\}) = 1 < 2 = \mathsf{I}_{\mathsf{MI}}(K \cup \{\beta\}).$$

On the other hand, let S_α and S_β be the scoring functions for $K \cup \{\alpha\}$ and $K \cup \{\beta\}$, respectively. Then

$$S_\alpha < S_\beta.$$

Contrary to the expectation about the MI inconsistency measure in [3,17], this example also shows that the measure I_{MI} is not a basic inconsistency measure.

The Free Formula Independence property emphasizes that free formulas of a belief base have nothing to do with inconsistency of that belief base, since free formulas do not belong to any minimal inconsistent subset of that belief base. Clearly this complies with the viewpoint that minimal inconsistent subsets are the purest form of inconsistency. But from the perspective of model-based or variable- based measuring for inconsistency, free formulas are not necessarily free from inconsistency, as illustrated by the example of $\{a \wedge \neg a \wedge b, \neg b\}$. So, Free Formula Independence is inappropriate for characterizing the model-based inconsistency measures. For example, it has been shown that a more general but representative variable-based measure $\mathsf{I}_{\mathsf{LP_m}}$ does not satisfy the Free Formula Independence property, since some free formula may increase the inconsistency value of a belief base [17].

As discussed earlier, for syntax sensitive proposals for measuring inconsistency for belief bases, each singleton set consisting of an individual free formula of a belief base is also considered as one of consistent plausible views of that belief base. In this sense, $\{a \wedge \neg a \wedge b, \neg b\}$ is different from $\{a \wedge \neg a \wedge b \wedge \neg b\}$ in a syntax sensitive application domain, because the former has a plausible view of $\{\neg b\}$, whereas the latter has no plausible view. Of course, adding free formulas to a belief base can enlarge the set of consistent subsets. But this cannot bring new minimal inconsistent subsets. Intuitively, adding or deleting free formulas may dilute or strengthen the degree of inconsistency. For example, as argued above, the scoring function implicitly supports this intuition in some sense. Furthermore, if adding a free formula of a belief base may weaken the degree of inconsistency of that belief base, then the Monotony property also does not hold.

To address the inappropriateness of the properties of Free Formula Independence, Dominance, and Monotony, we propose some new properties appropriate to characterizing the syntax-based inconsistency measure below.

4. A general framework for measuring the degree of inconsistency

From the perspective of syntax sensitive proposals for measuring inconsistency, each subset of a belief base provide a (partial) view of that belief base. Intuitively, each consistent subset conveys a distinctive plausible view of that belief base. Then it can be considered as one of basic views reflected by that belief base. On the other hand, the main idea of measuring inconsistency based on minimal inconsistent subsets is that minimal inconsistent subsets can be considered as the purest form of inconsistency from a syntax sensitive perspective, i.e., the inconsistency of a belief base is conveyed by only minimal inconsistent subsets of that belief base. From this perspective, each minimal inconsistent subset provides a distinctive atomic view of inconsistency in that belief base. So it can also be considered as one of basic views reflected by that belief base.

In contrast, any inconsistent but not minimal inconsistent subset conveys some consistent information as well as contradictory information. But with regard to contradictory information, any inconsistent but not minimal inconsistent subset provides no extra inconsistency other than that conveyed by its minimal inconsistent subsets. In this sense, any individual inconsistent but not minimal inconsistent subset of a belief base does not provide any new perspective of inconsistency in that belief base, except the perspectives of inconsistency reflected by the minimal inconsistent subsets of that inconsistent subset. According to this viewpoint, any inconsistent but not minimal inconsistent subset cannot be considered as a distinctive perspective of pure inconsistency. On the other hand, such an inconsistent subset brings no extra plausible views other than that conveyed by its consistent subsets. That is, it cannot also provide any new plausible view beyond basic consistent views reflected by that belief base. In summary, such an inconsistent subset conveys basic consistent views as well as basic inconsistent views. It can be considered as a mixed perspective generated by combining some consistent views and atomic inconsistent views. In this sense, only minimal inconsistent subsets and consistent subsets of a belief base can be considered as basic views reflected by that belief base. To illustrate this, consider $K = \{a, \neg a, c \wedge \neg c, d\}$ and $K' = \{a, \neg a, d\}$. Within the context of inconsistency measurements based on minimal inconsistent subsets, the inconsistency of K is characterized by $\mathsf{MI}(K) = \{\{a, \neg a\}, \{c \wedge \neg c\}\}$. As an inconsistent subset of K, the inconsistency of K' is captured by $\{a, \neg a\}$, one of the minimal inconsistent subsets of K. In other words, within the context of measuring inconsistency based on minimal inconsistent subsets, K' brings no new information other than the inconsistency conveyed by $\{a, \neg a\}$. Intuitively, K' conveys one distinct perspective of inconsistency, $\{a, \neg a\}$, as well as consistent distinct perspectives, $\{a\}$, $\{\neg a\}$, $\{d\}$, $\{a, d\}$, and $\{\neg a, d\}$.

Intuitively, how inconsistent a belief base is depends on the ratio of the inconsistent views to all the basic views reflected by that belief base. Hence, a syntax-based measure for the degree of inconsistency for a belief base in our framework should be based on the consistent subsets as well as minimal inconsistent subsets.

In order to integrate the impact of the size of a minimal inconsistent set on its degree of inconsistency and the impact of consistent subsets, we first give several auxiliary definitions.

Definition 4.1 (*k-size Minimal Inconsistent Subsets*). Let $\mathsf{MI}(K)$ be the set of minimal inconsistent subsets of K, then for each k $(1 \leq k \leq |K|)$, we define $\mathsf{MI}^{(k)}(K)$ (possibly empty) as the set of k-size minimal inconsistent subsets of K, i.e.,

$$\mathsf{MI}^{(k)}(K) = \{\Gamma \in \mathsf{MI}(K) | |\Gamma| = k\}.$$

Definition 4.2 (*k-size Consistent Subsets*). Let $\mathsf{CN}(K)$ be the set of consistent subsets of K, then for each k $(1 \leq k \leq |K|)$, we define $\mathsf{CN}^{(k)}(K)$ (possibly empty) as the set of k-size consistent subsets of K, i.e.,

$$\mathsf{CN}^{(k)}(K) = \{\Gamma \in \mathsf{CN}(K) | |\Gamma| = k\}.$$

In particular, we use $vcard(\mathsf{MI}(K))$ to denote the vector

$$(|\mathsf{MI}^{(1)}(K)|, \cdots, |\mathsf{MI}^{(|K|)}(K)|).$$

Note that $vcard(\mathsf{MI}(K))$ provides a more fine-grained characterization of the inconsistency of K than we have with the MI inconsistency measure. It considers the size of each minimal inconsistent subset of a belief base as well as the number of minimal inconsistent subsets with each size.

Example 4.1. Consider $K_5 = \{a \wedge \neg a, b, \neg b, c\}$. Then

$$\begin{aligned}
&\mathsf{MI}^{(1)}(K_5) = \{\{a \wedge \neg a\}\},\ \mathsf{MI}^{(2)}(K_5) = \{\{b, \neg b\}\},\ \mathsf{MI}^{(3)}(K_5) = \mathsf{MI}^{(4)}(K_5) = \emptyset.\\
&\mathsf{CN}^{(1)}(K_5) = \{\{b\}, \{\neg b\}, \{c\}\},\ \mathsf{CN}^{(2)}(K_5) = \{\{b, c\}, \{\neg b, c\}\},\\
&\mathsf{CN}^{(3)}(K_5) = \mathsf{CN}^{(4)}(K_5) = \emptyset.
\end{aligned}$$

Note that K_5 has one minimal inconsistent singleton subset and one 2-size minimal inconsistent subset. This can be reflected by $vcard(\mathsf{MI}(K_5)) = (1, 1, 0, 0)$. That is, in contrast to $|\mathsf{MI}(K)|$, $vcard(\mathsf{MI}(K))$ allows us to look inside the set of minimal inconsistent subsets of K.

Example 4.2. Consider $K_5 = \{a \wedge \neg a, b, \neg b, c\}$ again. Suppose that we add $b \wedge c$ to K_5, i.e., $K_5' = K_5 \cup \{b \wedge c\}$. Then

$$\begin{aligned}
&\mathsf{MI}^{(1)}(K_5') = \mathsf{MI}^{(1)}(K_5),\ \mathsf{MI}^{(2)}(K_5') = \mathsf{MI}^{(2)}(K_5) \cup \{\{b \wedge c, \neg b\}\},\\
&\mathsf{MI}^{(3)}(K_5') = \mathsf{MI}^{(4)}(K_5') = \mathsf{MI}^{(5)}(K_5') = \emptyset.\\
&\mathsf{CN}^{(1)}(K_5') = \mathsf{CN}^{(1)}(K_5) \cup \{\{b \wedge c\}\},\\
&\mathsf{CN}^{(2)}(K_5') = \mathsf{CN}^{(2)}(K_5) \cup \{\{b, b \wedge c\}, \{c, b \wedge c\}\},\\
&\mathsf{CN}^{(3)}(K_5') = \{b, b \wedge c, c\},\ \ \mathsf{CN}^{(4)}(K_5') = \mathsf{CN}^{(5)}(K_5') = \emptyset.
\end{aligned}$$

This indicates that adding some formulas may enlarge the set of minimal inconsistent subsets as well as the set of consistent subsets.

Definition 4.3 (*The Conflict Ratios Vector*)**.** Let K be a belief base, then the conflict ratios vector of K, denoted $\mathsf{CR}(K)$, is defined as

$$\mathsf{CR}(K) = (\mathsf{R}_1(K), \ldots, \mathsf{R}_{|K|}(K)),$$

where for each $i\,(1 \le i \le |K|)$,

$$\mathsf{R}_i(K) = \begin{cases} \frac{|\mathsf{MI}^{(i)}(K)|}{|\mathsf{MI}^{(i)}(K)|+|\mathsf{CN}^{(i)}(K)|}, & \text{if } |\mathsf{MI}^{(i)}(\mathrm{K})| + |\mathsf{CN}^{(i)}(\mathrm{K})| > 0;\\ 0, & \text{if } |\mathsf{MI}^{(i)}(\mathrm{K})| + |\mathsf{CN}^{(i)}(\mathrm{K})| = 0. \end{cases}$$

Note that for each $i\,(1 \le i \le |K|)$, $\mathsf{R}_i(K)$ is the ratio of the number of the i-size minimal inconsistent subsets to the sum of the number of the i-size minimal inconsistent subsets and the number of the i-size consistent subsets. Informally speaking, it may be considered as a normalized measure of k-size conflicts of K.

We use the following examples to illustrate the behavior of the conflict ratios vector.

Example 4.3 (*Continued*)**.** Consider K_5 and K_5' again. Then

$$\mathsf{CR}(K_5) = \left(\frac{1}{4}, \frac{1}{3}, 0, 0\right), \quad \mathsf{CR}(K_5') = \left(\frac{1}{5}, \frac{1}{3}, 0, 0\right).$$

Note that $\frac{1}{5} < \frac{1}{4}$. So this shows that adding $b \wedge c$ to K_5 brings more consistent information than inconsistent information, although it enlarges both the set of minimal inconsistent subset and the set of consistent subsets. It also implies that the conflict ratios vector does not support the Monotony property.

Example 4.4. Consider $K_6 = \{a \wedge \neg a\}$. K_6 is a minimal inconsistent singleton belief base. There is no consistent view reflected by K_6. In this case the conflict ratios vector is $\mathsf{CR}(K_6) = (1)$.

Example 4.5. Consider $K_7 = \{a, \neg a\}$. Note that K_7 is a minimal inconsistent belief base, and each 1-size subset of K_7 is consistent. Then the conflict ratios vector is $\mathsf{CR}(K_7) = (0, 1)$.

Example 4.6. Consider $K_8 = \{a, \neg a, b\}$. Note that b is a free formula of K_8. Moreover, K_8 has one 2-size minimal inconsistent subset and two 2-size consistent subsets. Hence the conflict ratios vector is $\mathsf{CR}(K_8) = (0, \frac{1}{3}, 0)$. Compared to the conflict ratios vector of K_7, adding the free formula b weakens the degree of 2-size confliction of K_7, since $\frac{1}{3} < 1$.

The last four examples illustrate that the conflict ratios vector is syntax sensitive.

Example 4.7. Consider $K_9 = \{a \wedge \neg a \wedge b, \neg b\}$. Note that K_9 has one minimal inconsistent singleton subset and one consistent singleton subset. Then the conflict ratios vector is $\mathsf{CR}(K_9) = (\frac{1}{2}, 0)$.

Example 4.8. Consider $K_{10} = \{a \wedge \neg a \wedge b \wedge \neg b\}$. This is a minimal inconsistent singleton belief base. Then the conflict ratios vector is $\mathsf{CR}(K_{10}) = (1)$.

Example 4.9. Consider $K_{11} = \{a \wedge \neg a, b \wedge \neg b\}$. Note that K_{11} has no consistent subset. Then the conflict ratios vector is $\mathsf{CR}(K_{11}) = (1, 0)$.

Example 4.10. Consider $K_{12} = \{a, \neg a, b, \neg b\}$. Note that K_{12} has two 2-size minimal inconsistent subsets and four 2-size consistent subsets. Here the conflict ratios vector is $\mathsf{CR}(K_{12}) = (0, \frac{1}{3}, 0, 0)$.

The conflict ratios vector of a belief base considers the number of minimal inconsistent subsets and consistent subsets as well as the size of each consistent or minimal inconsistent subset of that base. It may be used to characterize the degree of inconsistency for belief bases. In particular, as shown by the following proposition, the conflict ratios vector can describe succinctly the consistent belief base, the minimal inconsistent belief base, and the belief base without consistent subsets, respectively.

Proposition 4.1. *Let K be a belief base and let $\mathsf{CR}(K)$ be the conflict ratios vector of K.*

(1) $\mathsf{CR}(K) = \vec{0}$ *if and only if K is consistent, where $\vec{0}$ is the zero vector.*
(2) $\mathsf{R}_i(K) = 0$ *for each $i < |K|$ and $\mathsf{R}_{|K|}(K) = 1$ if and only if K is a minimal inconsistent belief base.*
(3) $\mathsf{R}_i(K) = 0$ *for each $i > 1$ and $\mathsf{R}_1(K) = 1$ if and only if none of non-empty subsets of K is consistent.*
(4) $\mathsf{R}_1(K) = 1$ *implies that $\mathsf{R}_i(K) = 0$ for each i ($2 \leq i \leq |K|$).*

Proof. Let K be a belief base and $\mathsf{CR}(K)$ the conflict ratios vector of K.

(1) $\mathsf{CR}(K) = \vec{0} \Longleftrightarrow$ for each k $(1 \leq k \leq |K|)$, $|\mathsf{MI}^{(i)}(K)| = 0 \Longleftrightarrow \mathsf{MI}(K) = \emptyset \Longleftrightarrow K$ is consistent.
(2) $\mathsf{R}_i(K) = 0$ for each $i < |K|$ and $\mathsf{R}_{|K|}(K) = 1 \Longleftrightarrow \mathsf{MI}(K) = \mathsf{MI}^{(|K|)}(K) = \{K\} \Longleftrightarrow K$ is a minimal inconsistent belief base.
(3) $\mathsf{R}_i(K) = 0$ for each $i > 1$ and $\mathsf{R}_1(K) = 1 \Longleftrightarrow \mathsf{MI}(K) = \mathsf{MI}^{(1)}(K) = \{\{\alpha\} | \alpha \in K\} \Longleftrightarrow \forall \alpha \in K, \alpha \vdash \perp$, i.e., none of non-empty subsets of K is consistent.
(4) If $\mathsf{R}_1(K) = 1$, then $\forall \alpha \in K, \{\alpha\} \vdash \perp$. Therefore, $|\mathsf{MI}^{(i)}(K)| = 0$ for each i $(2 \leq i \leq |K|)$. According to the definition of conflict ratios vector, $\mathsf{R}_i(K) = 0$ for each i $(2 \leq i \leq |K|)$. $\square$

Clearly, according to this proposition, if K is inconsistent, then there exists an i such that $1 \leq i \leq |K|$ and $\mathsf{R}_i(K) > 0$. Moreover, the conflict ratios vector of a belief base gives an overview of (normalized) conflicts with each size of that base. In this sense, the degree of inconsistency of a belief base K is captured by the value of each element of $\mathsf{CR}(K)$ together with the location of that element in $\mathsf{CR}(K)$. Then a normalized measure for the degree of inconsistency for K based on minimal inconsistent subsets should be an integrated function of the conflict ratios vector.

To provide a more general framework for measuring the degree of inconsistency of a belief base by using its minimal inconsistent subsets and consistent subsets, we define a measure function as follows:

Definition 4.4 (*Measure Functions*)**.** A measure function is a total function f_n associating a real number to every finite tuple of real numbers in [0, 1] and satisfying the following conditions:

(C1) $0 \leq f_n(x_1, \ldots, x_n) \leq 1$.
(C2) $f_n(x_1, \ldots, x_n) = 1$ if and only if $x_1 = 1$.
(C3) $f_n(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$.
(C4) $f_n(x_1, \ldots, x, \ldots, x_n) \leq f_n(x_1, \ldots, y, \ldots, x_n)$ if $x \leq y$.
(C5) $\lim_{n \to +\infty} f_n(0, \ldots, 0, 1) = 0$.
(C6) $f_n(0, \cdots, 0, 1) > f_{n+1}(0, \ldots, 0, 1)$.
(C7) $f_n(x_1, \ldots, x_n) = f_{n+1}(x_1, \ldots, x_n, 0)$.

Syntactically, a measure function is similar to the aggregation function defined in [22] in terms of a single value associated with a set of values, although they have different semantic interpretations. Roughly speaking, (C1) states that the measure function is a normalized function. (C2) and (C3) provide necessary and sufficient conditions on the lower bound 0 and the upper bound 1 of the measure function, respectively. (C4) requires that the measure function should be monotonic with regard to each variable. (C5) and (C6) provide constraints about the special function values $f_n(0, \ldots, 0, 1)$ and $f_m(0, \ldots, 0, 1)$. As illustrated by Proposition 4.1, n-tuple conflict ratios vector $(0, \ldots, 0, 1)$ corresponds to an n-size minimal inconsistent belief base. So, (C5) and (C6) may be considered as constraints on the degree of inconsistency of the minimal inconsistent belief base. Finally, (C7) provides a relation between the n-ary measure function and the $(n+1)$-ary measure function.

Based on measure functions, we now define measures of the degree of inconsistency for belief bases as follows.

Definition 4.5 (*Measures of the Degree of Inconsistency)***.** Let K be a belief base and $f_{|K|}$ a $|K|$-ary measure function. Then a measure of the degree of inconsistency for K induced by $f_{|K|}$, denoted $\mathsf{IncD}_{\mathsf{f}}(K)$, is defined as

$$\mathsf{IncD}_{\mathsf{f}}(K) = f_{|K|}(\mathsf{CR}(K)),$$

where $\mathsf{CR}(K)$ is the conflict ratios vector of K.

The choice of $f_{|K|}$ results in various instances of $\mathsf{IncD}_{\mathsf{f}}$. Moreover, as illustrated later, conditions (C1)–(C7) render the measures of the degree of inconsistency $\mathsf{IncD}_{\mathsf{f}}$ more intuitive.

5. Logical properties

Informally speaking, a rational set of properties to characterize inconsistency measures should focus on three aspects, namely natural constraints of inconsistency, natural constraints of inconsistency changes, and special characteristics of measures.

Generally, natural constraints of inconsistency describe the static nature of inconsistency, including how to distinguish consistent belief bases from inconsistent ones, whether the inconsistency measure is bounded, and the meaning of the bounds of the bounded measure. That is, natural constraints of inconsistency should include at least the following properties:

- *Characterization of consistent belief bases*: All the consistent belief base has the same value. Moreover, this value is different to any value for inconsistent belief base. For example, the Consistency property presented in [2] is such a property.
- *Characterization of bounds of inconsistency measurement*: For simplicity, inconsistency measures often need to be bounded, perhaps through normalization. With respect to this, the lower bound and the upper bound should have clear meanings or explicit explanations, respectively. For example, as mentioned above, in the context of maximal η-consistency, maximal 1-consistency corresponds to complete consistency, and maximal 0-consistency corresponds to the explicit presence of a contradiction, i.e., the explicit presence of a contradictory formula in a belief base [7]. Generally, if we assume that the bigger the inconsistency value is, the more inconsistent a belief base is, then the lower bound should be the value for consistent bases. In this sense, the Consistency property explains the meaning of the lower bound 0 for normalized inconsistency measures. Intuitively, the upper bound should be the value for the belief bases conveying only inconsistencies with regard to a given context of characterization of inconsistency. For example, to accord with this intuition, for a syntax sensitive measure I, $\mathsf{I}(K) = 1$ should imply that none of the non-empty subsets of K is consistent. In contrast, for a model-based or variable-based measure I, $\mathsf{I}(K) = 1$ should imply that all the variables of formulas in K can be given only the inconsistent truth value in paraconsistent models such as LP_m models.

Natural constraints on inconsistency changes characterize how inconsistency changes due to various changes of a belief base. For example, the properties of Monotony, Free Formula Independence and Dominance presented in [2] are such properties. Generally, we need to characterize the cases of strengthening inconsistency, weakening inconsistency, and unchanging inconsistency.

Special characteristics of special measures always make a particular type of measure distinctive. For example, for measures based on minimal inconsistent subsets, in addition to the two aspects above, we also need to take account of the special characteristics of minimal inconsistent subsets when considering the degree of inconsistency.

With this in mind, we now propose the following properties to characterize the inconsistency based on minimal inconsistent subsets.

First, we consider the natural constraints of inconsistency. Recall the five properties presented to characterize a basic inconsistency measure in [2,3], *Consistency* can be considered as one of the most intuitive constraints for any kind of inconsistency measure. So we should at least adopt the properties of *Consistency* and *Normalization* presented in [2] to characterize a normalized measure for the degree of inconsistency of a belief base. That is, let IncD be a measure for the degree of inconsistency of a belief base, IncD should satisfy

(P1) *Normalization*: $0 \leq \mathsf{IncD}(K) \leq 1$.
(P2) *Consistency*: $\mathsf{IncD}(K) = 0$ if and only if K is consistent.

As we discussed above, we need a new property to explain the meaning of the upper bound 1 for the normalized measure.

(P3) *Contradiction*: $\mathsf{IncD}(K) = 1$ if and only if K has no consistent subset.

The Contradiction property requires that a desirable inconsistency measure should assign the maximum degree of inconsistency to belief bases which have no consistent subsets. That is, if each singleton subset of a belief base is inconsistent, then that belief base should be considered as one of the most inconsistent belief bases. In particular, it also implies that the degree of inconsistency of any 1-size minimal inconsistent belief base (i.e. inconsistent singleton belief base) is the upper bound, that is 1.

In contrast, as argued above, *Free Formulas Independence*, *Dominance*, and *Monotony* are not appropriate for characterizing a normalized measure for the degree of inconsistency. Instead, we provide properties of *Free Formula Dilution* and *Monotony w.r.t. Conflict Ratio* below as natural constraints of inconsistency changes. Furthermore, we need to consider the special characteristics of the degree of inconsistency of a minimal inconsistent belief base, e.g., *Attenuation* and *Almost Consistency*. So a measure for the degree of inconsistency should also satisfy the following properties:

(P4) *Free Formula Dilution*: If $\alpha \in \mathsf{FREE}(K \cup \{\alpha\})$, then $\mathsf{IncD}(K \cup \{\alpha\}) \leq \mathsf{IncD}(K)$.
(P5) *Monotony w.r.t. Conflict Ratio*: If $|K'| = |K|$ and $\mathsf{R}_i(K) \leq \mathsf{R}_i(K')$ for each $1 \leq i \leq |K|$, then $\mathsf{IncD}(K) \leq \mathsf{IncD}(K')$.
(P6) *Attenuation*: $\forall M_1, M_2 \in \mathsf{MI}(K)$, $1 \geq \mathsf{IncD}(M_1) > \mathsf{IncD}(M_2) > 0$ if $|M_1| < |M_2|$.
(P7) *Almost Consistency*: $\lim_{|M| \to +\infty} \mathsf{IncD}(M) = 0$ for a minimal inconsistent belief base M.

The property of Free Formula Dilution states that adding or removing a free formula from a belief base can dilute or strengthen the degree of inconsistency of a belief base. The property of Monotony w.r.t. Conflict Ratio states that as the ratios of the number of i-size minimal inconsistent subsets to the number of i-size consistent or minimal inconsistent subsets increases, the measure of the degree of inconsistency cannot decrease. The property of Attenuation states that as the size of a minimal inconsistent subset increases, the degree of inconsistency becomes smaller. The property of Almost Consistency requires that as the size of a minimal inconsistent belief base increases, the degree of inconsistency tends to become 0, i.e., if the size of a minimal inconsistent belief base is large enough, it is nearly consistent. Note that the properties of Contradiction, Attenuation, and Almost Consistency are independent characteristics of the degree of inconsistency of a minimal inconsistent belief base. Each provides an intuitive restriction on the degree of a minimal inconsistent belief base.

Note that condition (C1) corresponds to the property (P1) (i.e., Normalization). Moreover, from Proposition 4.1, the special conflict ratios vectors $\vec{0}$, $(0, \ldots, 0, 1)$, and $(1, 0, \ldots, 0)$ correspond to the consistent belief base, the minimal inconsistent belief base, and the belief base without consistent subsets, respectively. (C2) and (C3) correspond to (P2) and (P3), respectively. Also, (C5) and (C6) correspond to (P7) and (P6), respectively. More generally, the following proposition shows that $\mathsf{IncD_f}$ is a desirable framework for measuring the degree of inconsistency of belief bases.

Proposition 5.1. $\mathsf{IncD_f}$ *satisfies the properties of Normalization, Consistency, Contradiction, Free Formula Dilution, Monotony w.r.t. Conflict Ratio, Attenuation, and Almost Consistency.*

Proof. Let K be a belief base.

(P1) *Normalization*: It follows from (C1) directly.
(P2) *Consistency*: From Proposition 4.1, K is consistent $\Leftrightarrow \mathsf{CR}(K) = \vec{0}$. According to (C3),

$$\mathsf{CR}(K) = \vec{0} \Leftrightarrow \mathsf{IncD_f}(K) = 0.$$

So, K is consistent $\Leftrightarrow \mathsf{IncD_f}(K) = 0$.
(P3) *Contradiction*: From Proposition 4.1, K has no consistent subsets, $\Leftrightarrow \mathsf{R}_1(K) = 1$ and $\mathsf{R}_i(K) = 0$ for $i > 1$. According to (C2),

$$\mathsf{CR}(K) = (1, 0, \ldots, 0) \;\Leftrightarrow\; \mathsf{IncD_f}(K) = 1.$$

So, K has no consistent subsets $\Leftrightarrow \mathsf{IncD_f}(K) = 1$.
(P4) *Free Formula Dilution*: If α is a free formula of $K \cup \{\alpha\}$, then

$$|\mathsf{MI}^{(i)}(K \cup \{\alpha\})| = \begin{cases} |\mathsf{MI}^{(i)}(K)|, & \text{if } 1 \leq i \leq |K|, \\ 0, & \text{if } i = |K| + 1. \end{cases}$$

and for each i ($1 \le i \le |K|$),

$$|\mathsf{MI}^{(i)}(K \cup \{\alpha\})| + |\mathsf{CN}^{(i)}(K \cup \{\alpha\})| \ge |\mathsf{MI}^{(i)}(K)| + |\mathsf{CN}^{(i)}(K)|.$$

So, for each $1 \le i \le |K|$,

$$\mathsf{R}_i(K \cup \{\alpha\}) \le \mathsf{R}_i(K),$$

and

$$\mathsf{R}_{|K|+1}(K \cup \{\alpha\}) = 0.$$

According to (C7),

$$f_{|K|+1}(\mathsf{R}_1(K \cup \{\alpha\}), \ldots, \mathsf{R}_{|K|}(K \cup \{\alpha\}), 0) = f_{|K|}(\mathsf{R}_1(K \cup \{\alpha\}), \ldots, \mathsf{R}_{|K|}(K \cup \{\alpha\})).$$

By iterated use of (C4),

$$f_{|K|}(\mathsf{R}_1(K \cup \{\alpha\}), \ldots, \mathsf{R}_{|K|}(K \cup \{\alpha\})) \le f_{|K|}(\mathsf{R}_1(K), \ldots, \mathsf{R}_{|K|}(K\})),$$

i.e., $\mathsf{IncD_f}(K \cup \{\alpha\}) \le \mathsf{IncD_f}(K)$.

(P5) *Monotony w.r.t. Conflict Ratio*: If $|K| = |K'|$ and $\mathsf{R}_i(K) \le \mathsf{R}_i(K')$ for each $1 \le i \le |K|$, then

$$f_{|K|}(\mathsf{R}_1(K), \ldots, \mathsf{R}_{|K|}(K\})) \le f_{|K|}(\mathsf{R}_1(K'), \ldots, \mathsf{R}_{|K|}(K'\})),$$

by iterated use of (C4). That is, $\mathsf{IncD_f}(K) \le \mathsf{IncD_f}(K')$.

(P6) *Attenuation*: From proposition 4.1, M is a minimal inconsistent belief base iff $\mathsf{R}_{|M|}(M) = 1$ and $\mathsf{R}_i(M) = 0$ for each $i < |M|$. According to (C6), $\mathsf{IncD_f}(M_1) > \mathsf{IncD_f}(M_2)$ if $|M_1| < |M_2|$. According to (P1) and (P2), $1 \ge \mathsf{IncD_f}(M_1) > \mathsf{IncD_f}(M_2) > 0$.

(P7) *Almost Consistency*: This can be deduced from (C5) and the proof for *Attenuation* directly. □

6. Instantiating the general framework

We now provide some simple measure functions to instantiate our framework. We attempt to take advantage of some current measures for the degree of inconsistency of minimal inconsistent belief bases such as maximal η-consistency to derive an instantiated measure for general belief bases.

Definition 6.1. A sequence of real numbers $D = \{d_i\}_{i=1}^{+\infty}$ is called a rudimentary sequence of the degree of inconsistency if it satisfies the following conditions:

(D1) $\forall n \in \mathbb{N}, d_n > 0$.
(D2) $d_1 = 1$.
(D3) $\forall n \in \mathbb{N}, d_{n+1} < d_n$.
(D4) $\lim_{n\to+\infty} d_n = 0$.

For example, both $\{\frac{1}{n}\}_{n=1}^{+\infty}$ and $\{e^{1-n}\}_{n=1}^{+\infty}$ are such sequences of real numbers. Note that (D2), (D3), and (D4) correspond to the properties of Contradiction, Attenuation, and Almost Consistency, respectively. Essentially, a rudimentary sequence of the degree of inconsistency is a sequence that accords with the intuitive constraints of the measures of the degree of inconsistency of each i-size minimal inconsistent belief base. In other words, it can be considered as a scheme to define the measure of the degree of inconsistency for minimal inconsistent belief bases.

Definition 6.2. Let $D = \{d_i\}_{i=1}^{+\infty}$ be a rudimentary sequence of the degree of inconsistency. The n-ary function based on D, denoted f_n^D, is defined as

$$f_n^D(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i \cdot d_i)\,.$$

Lemma 6.1. *Let $D = \{d_i\}_{i=1}^{+\infty}$ be a rudimentary sequence of the degree of inconsistency. Then f_n^D is a measure function, i.e., it satisfies* (C1)–(C7).

Proof. Let $D = \{d_i\}_{i=1}^{+\infty}$ be a rudimentary sequence of the degree of inconsistency.

(C1) $\forall(x_1, \ldots, x_n) \in [0, 1]^n, 0 \leq (1 - x_i \cdot d_i) \leq 1$. So,

$$0 \leq f_n^D(x_1, \ldots, x_n) \leq 1.$$

(C2) "$\Longrightarrow$:" If $x_1 = 1$, then $(1 - x_1 \cdot d_1) = 0$. So, $f_n^D(1, 0, \ldots, 0) = 1$.
"$\Longleftarrow$: " If $f_n^D(x_1, \ldots, x_n) = 1$, then $\prod_{i=1}^n (1 - x_i \cdot d_i) = 0$. According to (D3) and (D2), for each i $(2 \leq i \leq n)$, $1 - x_i \cdot d_i > 0$ since $d_i < d_1 = 1$. Therefore, $x_1 = 1$.
(C3) $f_n^D(x_1, \ldots, x_n) = 0 \Longleftrightarrow \prod_{i=1}^n (1 - x_i \cdot d_i) = 1 \Longleftrightarrow x_1 = \cdots = x_n = 0$.
(C4) if $x_k \leq y_k$ and $x_i = y_i$ for each $i \neq k$, then $\prod_{i=1}^n (1 - x_i \cdot d_i) \geq \prod_{i=1}^n (1 - y_i \cdot d_i)$. So, $1 - \prod_{i=1}^n (1 - x_i \cdot d_i) \leq 1 - \prod_{i=1}^n (1 - y_i \cdot d_i)$, i.e.,

$$f_n^D(x_1, \ldots, x_n) \leq f_n^D(y_1, \ldots, y_n).$$

(C5) $f_n(0, \ldots, 0, 1) = d_n$. According to (D4),

$$f_n^D(0, \ldots, 0, 1) = d_n \to 0 \text{ if } n \to +\infty.$$

(C6) $f_{n+1}(0, \ldots, 0, 1) = d_{n+1}$. According to (D3),

$$f_{n+1}^D(0, \ldots, 0, 1) = d_{n+1} < d_n = f_n^D(0, \ldots, 0, 1).$$

(C7) $f_{n+1}^D(x_1, \ldots, x_n, 0) = 1 - \prod_{i=1}^n (1 - x_i \cdot d_i) \cdot (1 - 0 \cdot d_{n+1}) = 1 - \prod_{i=1}^n (1 - x_i \cdot d_i)$. That is

$$f_{n+1}^D(x_1, \ldots, x_n, 0) = f_n^D(x_1, \ldots, x_n). \quad \square$$

Correspondingly, the inconsistency measure induced by f_n^D can be defined as follows.

Definition 6.3. Let $D = \{d_i\}_{i=1}^{+\infty}$ be a rudimentary sequence of the degree of inconsistency. Let K be a belief base. The measure for the degree of inconsistency of K, denoted $\mathsf{IncD}_{\mathsf{f^D}}(K)$, is given as follows:

$$\mathsf{IncD}_{\mathsf{f^D}}(K) = f_{|K|}^D(\mathsf{CR}(K)) = 1 - \prod_{i=1}^{|K|} (1 - \mathsf{R}_i(K) \cdot d_i)\,.$$

As discussed above, the degree of inconsistency of K is captured by the conflict ratios vector $(\mathsf{R}_1(K), \ldots, \mathsf{R}_{|K|}(K))$. The measure $\mathsf{IncD}_{\mathsf{f^D}}(K)$ provides a scheme to integrate these ratios, in which weight d_i reflects the relative location of $\mathsf{R}_i(K)$ in the ratios vector $(\mathsf{R}_1(K), \ldots, \mathsf{R}_{|K|}(K))$ for each i in a natural way. Moreover, the corresponding function f_n^D is not a simple weighted sum function. It provides a mechanism for weighted integration under the constraint of normalization.

As a direct consequence of Proposition 5.1 and Lemma 6.1, the following corollary shows that $\mathsf{IncD}_{\mathsf{f^D}}$ is an intuitive instantiated measure of the degree of inconsistency for belief bases.

Corollary 6.1. $\mathsf{IncD}_{\mathsf{f^D}}$ *satisfies the properties of Normalization, Consistency, Contradiction, Free Formula Dilution, Monotony w.r.t. Conflict Ratio, Attenuation, and Almost Consistency.*

In particular, for any minimal inconsistent belief base M,

$$\mathsf{IncD}_{\mathsf{f^D}}(M) = d_{|M|}.$$

This is why we call $\{d_i\}_{i=1}^{+\infty}$ the rudimentary sequence of the degree of inconsistency. Then we may use some current measures for the degree of inconsistency for minimal inconsistent belief bases to further instantiate $\mathsf{IncD}_{\mathsf{f^D}}$.

As discussed earlier, the degree of inconsistency of the belief base K is characterized by the value of each element of the conflict ratios vector $\mathsf{CR}(K)$ together with the location of that element in $\mathsf{CR}(K)$. We are interested in the issue that whether this characterization can be captured by this instantiated measure $\mathsf{IncD}_{\mathsf{f^D}}$. To illustrate this, consider the following example.

Example 6.1. Consider $K = \{a, \neg a, b, c, d\}$ and $K' = \{a, a \to b, \neg b, c, d\}$. Then $\mathsf{MI}(K) = \{\{a, \neg a\}\}$ and $\mathsf{MI}(K') = \{\{a, a \to b, \neg b\}\}$.

Note that neither $\mathsf{I_{MI}}$ nor normalized $\mathsf{I_{MI}}$ can make a distinction between K and K', since

$$\mathsf{I_{MI}}(K) = \mathsf{I_{MI}}(K') = 1$$

and

$$\frac{\mathsf{I}_{\mathsf{MI}}(\mathsf{K})}{2^{|K|}-1} = \frac{\mathsf{I}_{\mathsf{MI}}(K')}{2^{|K'|}-1} = \frac{1}{31}.$$

Intuitively, K' is different from K in the degree of inconsistency from a syntax sensitive perspective. However, the conflict ratios vector can make a distinction between the two belief bases, because $\mathsf{CR}(K) = (0, \frac{1}{10}, 0, 0, 0)$ and $\mathsf{CR}(K') = (0, 0, \frac{1}{10}, 0, 0)$, although the two conflict ratios vectors have the common element $\frac{1}{10}$.

Moreover, the measure $\mathsf{IncD}_{\mathsf{fD}}$ can capture this distinction, since

$$\mathsf{IncD}_{\mathsf{fD}}(K) = \frac{d_2}{10} > \frac{d_3}{10} = \mathsf{IncD}_{\mathsf{fD}}(K').$$

More generally, the following proposition shows this instantiated measure provides direct support for this characterization.

Proposition 6.1. *Suppose that K_1 and K_2 are two n-size belief bases. Let $1 \le l < m \le n$ be two numbers such that*

(1) $\mathsf{R}_i(K_1) = \mathsf{R}_i(K_2)$ *if* $i \notin \{l, m\}$.
(2) $\mathsf{R}_l(K_1) = \mathsf{R}_m(K_2)$, $\mathsf{R}_m(K_1) = \mathsf{R}_l(K_2)$.
(3) $\mathsf{R}_l(K_1) \le \mathsf{R}_m(K_1)$.

Then

$$\mathsf{IncD}_{\mathsf{fD}}(K_1) \le \mathsf{IncD}_{\mathsf{fD}}(K_2).$$

Proof. Let $c = \prod_{1 \le i \le n, i \notin \{l,m\}} (1 - \mathsf{R}_i(K_1) \cdot d_i)$. According to (1) and (2),

$$\mathsf{IncD}_{\mathsf{fD}}(K_1) = 1 - c(1 - \mathsf{R}_l(K_1) \cdot d_l)(1 - \mathsf{R}_m(K_1) \cdot d_m),$$

$$\mathsf{IncD}_{\mathsf{fD}}(K_2) = 1 - c(1 - \mathsf{R}_m(K_1) \cdot d_l)(1 - \mathsf{R}_l(K_1) \cdot d_m).$$

Then

$$\begin{aligned}
&\mathsf{IncD}_{\mathsf{fD}}(K_2) - \mathsf{IncD}_{\mathsf{fD}}(K_1)\\
&\quad = c((1 - \mathsf{R}_l(K_1) \cdot d_l)(1 - \mathsf{R}_m(K_1) \cdot d_m) - (1 - \mathsf{R}_m(K_1) \cdot d_l)(1 - \mathsf{R}_l(K_1) \cdot d_m))\\
&\quad = c(d_l - d_m)(\mathsf{R}_m(K_1) - \mathsf{R}_l(K_1)).
\end{aligned}$$

Obviously, $c > 0$, and $(d_l - d_m) > 0$, since $l < m$. By (3), we can get

$$\mathsf{IncD}_{\mathsf{fD}}(K_2) - \mathsf{IncD}_{\mathsf{fD}}(K_1) \ge 0.$$

That is,

$$\mathsf{IncD}_{\mathsf{fD}}(K_1) \le \mathsf{IncD}_{\mathsf{fD}}(K_2). \quad \square$$

Informally speaking, this proposition shows that the instantiated measure focuses on the relative location of each element of the conflict ratios vector as well as these ratios. This accords with the characterization of inconsistency in terms of the conflict ratios vector.

The maximal η-consistency presented in [7] describes how consistent a belief base is. Let η_i be the value of η for the i-size minimal inconsistent subset, then we define a rudimentary sequence of the degree of inconsistency $D^\eta = \{d_i^\eta\}_{i=1}^{+\infty}$ such that $d_i^\eta = 1 - \eta_i = \frac{1}{i}$.

We then give the corresponding instantiated measure for the degree of inconsistency as follows:

Definition 6.4. Let K be a belief base. Then

$$\mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\eta}}(K) = 1 - \prod_{i=1}^{n}\left(1 - \mathsf{R}_i(K)\cdot\frac{1}{i}\right).$$

Example 6.2. Consider $K_6 = \{\neg a, a\}$, $K_7 = \{\neg a, a, b\}$ and $K_8 = \{\neg a, \neg b, a \vee b\}$ again. Note that $\mathsf{MI}(K_6) = \mathsf{MI}(K_7) = \{\{\neg a, a\}\}$ and $\mathsf{MI}(K_8) = \{K_8\}$. As discussed above, we cannot compare K_6 and K_7 in terms of scoring functions, since $|K_6| \neq |K_7|$. On the other hand, if we adopt the MI inconsistency measure I_{MI}, then

$$\mathsf{I}_{\mathsf{MI}}(K_6) = \mathsf{I}_{\mathsf{MI}}(K_7) = \mathsf{I}_{\mathsf{MI}}(K_8) = 1.$$

Here we cannot make a distinction among the inconsistencies of the three belief bases. Furthermore, if we consider the normalized MI inconsistency measure, then

$$\frac{\mathsf{I}_{\mathsf{MI}}(K_6)}{2^{|K_6|}-1} = \frac{1}{3}, \quad \frac{\mathsf{I}_{\mathsf{MI}}(K_7)}{2^{|K_7|}-1} = \frac{\mathsf{I}_{\mathsf{MI}}(K_8)}{2^{|K_8|}-1} = \frac{1}{7}.$$

Again we cannot make a distinction between the degree of inconsistency of K_7 and that of K_8. However, the scoring function-based comparison shows that K_8 is more inconsistent than K_7.

Now consider the measure $\mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\eta}}$, then

$$\mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\eta}}(K_6) = \frac{1}{2}, \quad \mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\eta}}(K_7) = \frac{1}{6}, \quad \mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\eta}}(K_8) = \frac{1}{3}.$$

With this measure, we can distinguish the three belief bases from each other in the sense of the degree of inconsistency, i.e, K_6 is more inconsistent than K_8, and K_8 is more inconsistent than K_7.

As discussed earlier, the normalized MI inconsistency measure $\frac{\mathsf{I}_{\mathsf{MI}}(K)}{2^{|K|}-1}$ cannot make a distinction in the degree of inconsistency between any two belief bases with the same size and the same number of minimal inconsistent subsets. However, if we focus on the minimal inconsistent belief bases, this normalized measure for minimal inconsistent belief bases satisfies the properties of Contradiction, Attenuation, and Almost consistency. Then we can also use the normalized MI inconsistency measures for minimal inconsistent belief bases to define a rudimentary sequence of the degree of inconsistency D^{I} as follows:

$$d_i^{\mathsf{I}} = \frac{\mathsf{I}_{\mathsf{MI}}(M)}{2^{|M|}-1},$$

where M is a minimal inconsistent belief base and $|M| = i$. Since $\mathsf{I}_{\mathsf{MI}}(M) = 1$, $d_i^{\mathsf{I}} = \frac{1}{2^i-1}$. Correspondingly, we give the instantiated measure for the degree of inconsistency as follows:

Definition 6.5. Let K be a belief base. Then

$$\mathsf{IncD}_{\mathsf{f}\mathsf{D}^{\mathsf{I}}}(K) = 1 - \prod_{i=1}^{n}\left(1 - \mathsf{R}_i(K)\cdot\frac{1}{2^i-1}\right).$$

Example 6.3. Consider $K_9 = \{a, \neg a, b, \neg b, c\}$ and $K_{10} = \{a, \neg a, \neg a \vee b, \neg b, c\}$. Then

$$\mathsf{MI}(K_9) = \{\{a, \neg a\}, \{b, \neg b\}\}, \ \mathsf{MI}(K_{10}) = \{\{a, \neg a\}, \{a, \neg a \vee b, \neg b\}\}.$$

If we adopt the MI inconsistency measure, then

$$\mathsf{I}_{\mathsf{MI}}(K_9) = \mathsf{I}_{\mathsf{MI}}(K_{10}) = 2.$$

Note that $|K_9| = |K_{10}|$, then

$$\frac{\mathsf{I}_{\mathsf{MI}}(K_9)}{2^{|K_9|}-1} = \frac{\mathsf{I}_{\mathsf{MI}}(K_{10})}{2^{|K_{10}|}-1} = \frac{2}{31}.$$

Therefore both the MI inconsistency measure and the normalized MI inconsistency measure cannot make a distinction between the two belief bases with respect to the degree of inconsistency.

In contrast, consider the measure $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\mathsf{I}}}}$, then

$$\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\mathsf{I}}}}(K_9) = \frac{1}{15} > \mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\mathsf{I}}}}(K_{10}) = \frac{13}{245}.$$

With this measure, we can distinguish the two belief bases from each other in the sense of the degree of inconsistency, i.e, K_9 is more inconsistent than K_{10}.

As shown by the following corollary, the two instances of $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}}}$ satisfy the expected properties.

Corollary 6.2. *Both* $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\eta}}}$ *and* $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\mathsf{I}}}}$ *satisfy the properties of Normalization, Consistency, Contradiction, Free Formula Dilution, Monotony w.r.t. Conflict Ratio, Attenuation, and Almost Consistency.*

As an instance of measure functions, f^{D_n} induces two instantiated inconsistency measures $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\eta}}}$ and $\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^{\mathsf{I}}}}$ based on two different but intuitive rudimentary sequences of the degree of inconsistency, respectively. Note that these two rudimentary sequence of the degree of inconsistency come from current inconsistency measures. In this sense, the two instantiated measures also take the best of the corresponding current measures.

7. An application in requirements engineering

Here we use an example in requirements engineering to illustrate the application of our measure of the degree of inconsistency for belief bases.

Example 7.1. Consider a scenario for eliciting requirements for updating an existing software system. Essentially the idea is that the requirements are refined by negotiation. We want to check if the refinements are improving from a consistency point of view.

At first, two stakeholders express their demands from their perspectives, respectively.

(a) Stakeholder A: A representative of the sellers of the new system, provides the following demands:
  (a1) The user interface of the system-to-be should be in the modern idiom (i.e., fashionable).
  (a2) The system-to-be should be developed based on the newest development techniques.
(b) Stakeholder B: A representative of the users of the existing system, provides the following demands:
  (b1) The system-to-be should be developed based on the techniques used in the existing system.
  (b2) The user interface of the system-to-be should maintain the style of the existing system.
  (b3) The system-to-be should be secure.
(c) The domain expert in requirements engineering provides the following constraint, which is a consequence of (b3) above:
  (c1) To guarantee the security of the system-to-be, openness (or ease of extension) should not be considered.

Suppose that we

- use the predicate Fash(int_f) to denote that the interface is fashionable;
- use the predicate Open(sys) to denote that the system is open;
- use the predicate New(sys) to denote that the system will be developed based on the newest techniques;
- use the predicate Secu(sys) to denote that the system is secure.

Then we use a belief base

$$K_{r0} = \{\mathrm{Fash(int\_f)}, \mathrm{New(sys)}, \neg\mathrm{Fash(int\_f)}, \\ \neg\mathrm{New(sys)}, \mathrm{Secu(sys)}, \mathrm{Secu(sys)} \to \neg\mathrm{Open(sys)}\}$$

to represent the requirements above, these are taken respectively in the order $\{(a), (b), (c)\}$. Clearly, the following inconsistencies can be identified from these requirements:

$$K_{r0} \vdash \mathrm{New(sys)} \wedge \neg\mathrm{New(sys)},$$
$$K_{r0} \vdash \mathrm{Fash(int\_f)} \wedge \neg\mathrm{Fash(int\_f)}.$$

To resolve the inconsistencies in K_{r0}, some requirements need to be abandoned or to be changed. Before negotiation with stakeholders aimed at resolving inconsistency, the developers establish how inconsistent the set of requirements is.

The set of all the minimal inconsistent subsets of K_{r0} is $\mathsf{MI}(K_{r0}) = \{\mathsf{M}_1, \mathsf{M}_2\}$, where

$$\mathsf{M}_1 = \{\text{New}(\text{sys}), \neg\text{New}(\text{sys})\}, \quad \mathsf{M}_2 = \{\text{Fash}(\text{int\_f}), \neg\text{Fash}(\text{int\_f})\}.$$

Then we get the degree of inconsistency of K_{r0} as follows:

$$\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^\eta}}(K_{r0}) = \frac{1}{15}.$$

To guide the negotiation process toward resolving inconsistency, we need to know whether each round of negotiation abates the inconsistencies in requirements. It makes measuring the degree of inconsistency in requirements after each round of negotiation more necessary.

Suppose that before negotiation, Stakeholder A wants to know whether the degree of inconsistency in requirements remains unchanged if he persists in extending requirement (a1) as follows:

(a1') The user interface of the system-to-be should be in the modern idiom, and the system-to-be should be open.

In this case, we use a belief base

$$\begin{aligned} K'_{r0} = \{&\text{Fash}(\text{int\_f}) \wedge \text{Open}(\text{sys}), \text{New}(\text{sys}), \neg\text{Fash}(\text{int\_f}), \\ &\neg\text{New}(\text{sys}), \text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg\text{Open}(\text{sys})\} \end{aligned}$$

to represent the possible requirements. Evidently, K'_{r0} is inconsistent and $\mathsf{MI}(K'_{r0}) = \{\mathsf{M}'_1, \mathsf{M}'_2, \mathsf{M}'_3\}$, where

$$\begin{aligned} \mathsf{M}'_1 &= \{\text{New}(\text{sys}), \neg\text{New}(\text{sys})\}, \\ \mathsf{M}'_2 &= \{\text{Fash}(\text{int\_f}) \wedge \text{Open}(\text{sys}), \neg\text{Fash}(\text{int\_f})\}, \\ \mathsf{M}'_3 &= \{\text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg\text{Open}(\text{sys}), \text{Fash}(\text{int\_f}) \wedge \text{Open}(\text{sys})\}. \end{aligned}$$

The degree of inconsistency of K'_{r0} is calculated as follows:

$$\mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^\eta}}(K'_{r0}) = \frac{5}{54} > \frac{1}{15} = \mathsf{IncD}_{\mathsf{f}^{\mathsf{D}^\eta}}(K_{r0}).$$

This implies that this possible extension may strengthen the degree of inconsistency of requirements. Stakeholder A recognizes that he needs to make some concession.

Suppose that after a round of negotiation, Stakeholder A agrees to abandon requirement (a1), and provides a new demand:

(a3) The system-to-be should be open, that is, the system-to-be could be extended easily.

Then we use a new belief base

$$\begin{aligned} K_{r1} = \{&\text{Open}(\text{sys}), \text{New}(\text{sys}), \neg\text{Fash}(\text{int\_f}), \\ &\neg\text{New}(\text{sys}), \text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg\text{Open}(\text{sys})\} \end{aligned}$$

to represent the requirements after negotiation. Clearly, the new set of requirements is also inconsistent:

$$\begin{aligned} K_{r1} &\vdash \text{New}(\text{sys}) \wedge \neg\text{New}(\text{sys}), \\ K_{r1} &\vdash \text{Open}(\text{sys}) \wedge \neg\text{Open}(\text{sys}). \end{aligned}$$

The developers would like to know whether the new requirements are less inconsistent than the originals. The set of all the minimal inconsistent subsets of K_{r1} is $\mathsf{MI}(K_{r1}) = \{\mathsf{M}_1, \mathsf{M}_3\}$, where

$$\begin{aligned} \mathsf{M}_1 &= \{\text{New}(\text{sys}), \neg\text{New}(\text{sys})\}, \\ \mathsf{M}_3 &= \{\text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg\text{Open}(\text{sys}), \text{Open}(\text{sys})\}. \end{aligned}$$

And the degree of inconsistency of K_{r1} is calculated as follows:

$$\mathrm{IncD}_{\mathrm{fD}^{\eta}}(K_{r1}) = \frac{77}{1440} < \frac{1}{15} = \mathrm{IncD}_{\mathrm{fD}^{\eta}}(K_{r0}).$$

That is, after the first round of negotiation, the requirements have become less inconsistent. This in turn signifies informally that this negotiation made some progress.

Suppose that the developers proceed with a second negotiation, and Stakeholder B agrees to abandon requirement (b2). Then we use the new belief base

$$K_{r2} = \{\mathrm{Open(sys)}, \mathrm{New(sys)}, \neg\mathrm{Fash(int\_f)}, \mathrm{Secu(sys)}, \mathrm{Secu(sys)} \to \neg\mathrm{Open(sys)}\}$$

to represent the requirements after this negotiation. Clearly, the new set of requirements is also inconsistent:

$$K_{r2} \vdash \mathrm{Open(sys)} \wedge \neg\mathrm{Open(sys)}.$$

The set of all the minimal inconsistent subsets of K_{r2} is $\mathrm{MI}(K_{r2}) = \{\mathrm{M}_3\}$, where

$$\mathrm{M}_3 = \{\mathrm{Secu(sys)}, \mathrm{Secu(sys)} \to \neg\mathrm{Open(sys)}, \mathrm{Open(sys)}\}.$$

The degree of inconsistency of K_{r2} is calculated this time as follows:

$$\mathrm{IncD}_{\mathrm{fD}^{\eta}}(K_{r2}) = \frac{1}{30} < \mathrm{IncD}_{\mathrm{fD}^{\eta}}(K_{r1}).$$

This signifies that the second round of negotiation also makes progress.

Suppose that after a third round of negotiation, Stakeholder A agrees to withdraw requirement (a3), then

$$K_{r3} = \{\mathrm{New(sys)}, \neg\mathrm{Fash(int\_f)}, \mathrm{Secu(sys)}, \mathrm{Secu(sys)} \to \neg\mathrm{Open(sys)}\}$$

and

$$\mathrm{IncD}_{\mathrm{fD}^{\eta}}(K_{r3}) = 0.$$

This means the inconsistency in the original requirements has been resolved by negotiation.

However, if we adopt the MI inconsistency measure, then after the first negotiation,

$$\mathrm{I}_{\mathrm{MI}}(K_{r0}) = \mathrm{I}_{\mathrm{MI}}(K_{r1}) = 2.$$

This serves as an illustration that the MI inconsistency measure can not reflect the change of the degree of inconsistency of requirements after requirements negotiation. As discussed earlier, the MI inconsistency measure focused on only the total number of minimal inconsistent subsets. This implies that the MI inconsistency measure cannot capture the possible change of inconsistency due to the change of size of the minimal inconsistent subset in the cases such as the first round negotiation.

In contrast, our inconsistency measure attempts to capture the impact of consistent subsets as well as that of the size of each minimal inconsistent subset and the number of minimal inconsistent subsets on the degree of inconsistency for belief bases. It is sensitive to either changes of consistent subsets or changes of minimal inconsistent subsets due to syntax changes of a belief base such as augmenting a belief base, or removing some formulas from a belief base, or replacing some original formulas with new formulas. So, it is more appropriate to capture changes of the degree of inconsistency in requirements negotiation above.

8. Comparison and discussion

In this section we compare the properties and the measures for the degree of inconsistency for belief bases with some closely related research works.

The underlying principle of measures defined in this paper is to define measures for the degree of inconsistency of a belief base by using its minimal inconsistent subsets and consistent subsets. However, deriving an inconsistency measure for a belief base from its minimal inconsistent subsets has been increasingly recognized as a natural way to articulate the inconsistency of a belief base [14,3]. This is in accord with the viewpoint of minimal inconsistent subsets as the purest form of inconsistency [15]. Moreover, as argued in [3], such a measure is necessary for some syntax sensitive applications such as formal reasoning about software requirements.

How to characterize a desirable inconsistency measure especially an inconsistency measure based on minimal inconsistent subsets is still under development in the research community. Roughly speaking, a rational set of postulates or properties to characterize a special type of inconsistency measure needs to consider the special features for the special type of measures as well as general constraints for any desirable inconsistency measures. Regarding the set of properties presented in this paper, the Consistency property and the Normalization property come from the set of five properties that are claimed to characterize a basic inconsistency measure in [2,3,17]. These two common properties can be considered as natural and intuitive constraints about the degree of inconsistency of belief bases. On the other hand, the Normalization property creates a demand for the Contradiction property, which explains explicitly the meaning of the upper bound 1 of the measures of the degree of inconsistency in terms of minimal inconsistent subsets and consistent subsets. Moreover, the properties of Contradiction, Attenuation, and Almost Consistency together provide a special characterization of the degree of inconsistency for minimal inconsistent subsets. It is crucial to further characterize inconsistency measures based on minimal inconsistent subsets. In contrast, the set of five properties presented in [2,3] lacks an explicit characterization about the degree of inconsistency for minimal inconsistent subsets. We proposed the properties of Free Formula Dilution and Monotony w.r.t. Conflict Ratio instead of the properties of Free Formula Independence and Monotony presented in [2,3] to articulate changes of the degree of inconsistency due to changes of belief bases, respectively. As argued earlier, the impact of free formulas on the degree of inconsistency cannot be captured by the Free Formula Independence property either in the case of syntax-based measures such as the scoring function [14] or in the case of model-based measures such as $\mathsf{I}_{\mathsf{LP_m}}$ [17]. Intuitively, for the inconsistency measures based on minimal inconsistent subsets, adding free formulas to a belief base may dilute the degree of inconsistency of that base, since it enlarges the set of consistent subsets of that belief base. Clearly, the Monotony property also contradicts this intuition.

Note that the Contradiction property states that any belief base without consistent subsets has the degree of inconsistency at the upper bound. This complies with the viewpoint that each consistent subset of a belief base is viewed as a plausible perspective of that belief base. It is really syntax sensitive. To illustrate this, consider $\{a \wedge \neg a \wedge b\}$ and $\{a \wedge \neg a, b\}$. $\mathsf{Inc}(\{a \wedge \neg a \wedge b\}) = 1 > \mathsf{Inc}(\{a \wedge \neg a, b\})$, since the latter has one plausible consistent view $\{b\}$.

Under the guidance of these properties, we presented a general framework and some instances of this framework to measure the degree of inconsistency of a belief base by using its minimal inconsistent subsets and consistent subsets. The scoring function presented in [14] and the MI inconsistency measure presented in [3] are most closely related to our measures proposed in this paper. The two proposals based on minimal inconsistent subsets consider the number of minimal inconsistent subsets of a belief base as an essential factor to define an inconsistency measure. As mentioned earlier, the scoring function uses $2^{|K|}$ values to articulate the degree of inconsistency of K. This renders the scoring function difficult to use when comparing any two inconsistent belief bases with different sizes. The MI inconsistency measure considers the number of minimal inconsistent subsets as the amount of inconsistency of a belief base. However, both the scoring function and the MI inconsistency measure do not consider the impact of the size of each minimal inconsistent belief base on the degree of inconsistency of that belief base explicitly. So, neither the MI inconsistency measure nor the normalized the MI inconsistency measure makes a distinction between any two belief bases with the same size and the same number of minimal inconsistent subsets.

In contrast, there are a number of characteristics of the measures defined in this paper distinguish our measures from the scoring function and the MI inconsistency measure. First, the general framework proposed in this paper is based on the conflict ratios vector of a belief base rather than on the minimal inconsistent subset directly. The conflict ratios vector of a belief base considers the size of each minimal inconsistent subset as well as the number of minimal inconsistent subsets, but also considers the size and the number of consistent subsets. Second, the satisfaction of Contradiction, Attenuation, and Almost Consistency implies that our general framework and the corresponding instances of measures for the degree of inconsistency support the intuition illustrated by the lottery paradox that the degree of inconsistency decreases as the size of a minimal inconsistent subset increases. In this sense, it accords with the strict n-consistency presented in [9] and the maximal η-consistency presented in [7] about the degree of inconsistency of minimal inconsistent belief bases. Third, the satisfaction of the Free Formula Dilution property signifies that the impact of free formulas on the degree of inconsistency of a belief base can be captured by the corresponding instantiated measures. There are many different ways of instantiating function D to define different IncD functions. This will be the main direction of our future work.

With regard to potential implementation of inconsistency measures presented in this paper, the core is to compute minimal inconsistent subsets and consistent subsets for a belief base. One of underlying problems of the core is to check whether a set of formulas is consistent or not, i.e., a SAT problem. This makes the core computationally hard, since SAT is NP-complete [23], and checking whether a set of clauses is a minimal inconsistent subset or not is DP-complete [24]. However, as pointed out in [3], the impressive progress in SAT solvers in recent years has promoted techniques for practically identifying minimal inconsistent subsets of a belief base. Some algorithms such as [25] have been proposed to practically find each minimal inconsistent subset (called Minimally Unsatisfiable Subformulas or MUS in these algorithms) of a belief base. On the other hand, as an efficient library of SAT solvers in Java, SAT4J library 2 [1] facilitates the first-time users of SAT ¡°black

[1] https://wiki.objectweb.org/sat4j/.

boxes¡±, who want to embed SAT technologies into their application without worrying about the details.[2] In future work, we will implement a prototype tool to measure inconsistency by making use of SAT4J library as well as the existing algorithm presented in [25].

9. Conclusion

Viewing minimal inconsistent subsets as the purest form of inconsistency, it is natural to derive syntax sensitive inconsistency measures for a belief base from the minimal inconsistent subsets of that belief base.

We have presented a framework for measuring the degree of inconsistency of a belief base by using its minimal inconsistent subsets along with its consistent subsets. It allows us to consider the impact of the size of each minimal inconsistent subset on the degree of inconsistency as well as to well characterize the role of free formulas in measuring the degree of inconsistency.

We have argued that the number of minimal inconsistent subsets of a belief base is insufficient to capture the degree of inconsistency of that belief base. Moreover, the Free Formula Independence property does not well characterize the role of free formulas in measuring the degree of inconsistency from a syntax sensitive perspective. Then Free Formula Dilution property instead of Free Formula Independence was proposed to characterize the change of the degree of inconsistency of a belief base due to adding or removing free formulas from that belief base. Furthermore, Contradiction, Attenuation, and Almost Consistency were proposed as the essential properties to characterize the degree of inconsistency of each minimal inconsistent subset. Motivated by these intuitive properties, we presented a general framework to define normalized measures for the degree of inconsistency of a belief base from its minimal inconsistent subsets along with its consistent subsets. We also have shown the measure defined from this general framework satisfies all the properties presented in this paper to characterize an intuitive measure of the degree of inconsistency for belief bases. Finally, we presented two instantiated measures based on two different but intuitive rudimentary sequences of the degree of inconsistency. To illustrate the practical potential usage of our measures, we presented a software engineering application.

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